Treeable CBERs are classifiable by ℓ_1

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Background and context

Invariant descriptive set theory is the study of definable equivalence relations and reductions between them.

A Polish space is a separable topological space with a compatible complete metric.

A Borel equivalence relation on a Polish space X is an equivalence relation on X which is Borel as a subset of $X \times X$.

Background and context

A Borel reduction from an equivalence relation E living on Polish X to an equivalence relation F living on Polish Y is a Borel function $f : X \to Y$ satisfying $x\ E\ y$ if and only if $f(x)\ F\ f(y)$.

By orbit equivalence relation, we will be referring to those that are induced by continuous actions of Polish groups on Polish spaces.

Say that an equivalence relation is classifiable by a Polish group G if and only if it is Borel reducible to an orbit equivalence relation induced by G.

Background and context

A Borel equivalence relation is called countable (resp. finite) if every class is countable (resp. finite)

Feldman-Moore: every countable Borel equivalence relation (CBER) is (up to a change of compatible Polish topology) an orbit equivalence relation induced by a countable group.

A CBER E is hyperfinite if $E = \bigcup_n F_n$ where each F_n is a finite Borel equivalence relation.

Slaman-Steel: A CBER is hyperfinite if and only if classifiable by $\mathbb{Z}$.

Background and context

Theorem 1 (Gao-Jackson [GJ15])

Let Δ be a countable discrete abelian group and $\Delta \curvearrowright X$ a continuous action on a Polish space. Then E_X^Δ is hyperfinite.

Hjorth: must every CBER classifiable by an abelian Polish group be hyperfinite?

Background and context

Theorem 1 (Gao-Jackson [GJ15])

Let Δ be a countable discrete abelian group and $\Delta \curvearrowright X$ a continuous action on a Polish space. Then E_X^Δ is hyperfinite.

Hjorth: must every CBER classifiable by an abelian Polish group be hyperfinite?

Ding-Gao: Every CBER classifiable by a non-Archimedean abelian Polish group is hyperfinite [DG17]

Cotton: Every CBER classifiable by a locally-compact abelian Polish group is hyperfinite [Cot19]

RESULTS

Theorem 2 (A. [All])
If E is a treeable CBER then E is classifiable by an abelian Polish group (in particular, ℓ_1).

The free part of the Bernoulli shift of F_2 is treeable but not hyperfinite [JKL]. Thus the answer to Hjorth's question is no.

On the other hand:

Theorem 3 (A. [All])
Any CBER classifiable by $\mathbb{R}^\omega$ is hyperfinite.

The negative answer

If E is *treeable* that means there is a Borel tree T on X such that $E = E_T$. Up to Borel bi-reducibility we can assume the treeing is locally-finite [JKL02].

A *Polish edge labeling* is a Polish space L and a Borel injective function $\ell : L \to T$ such that for every $(x, y) \in T$, exactly one of (x, y) and (y, x) is in $\ell[L]$.

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Let $V(L)$ be the vector space generated by L, extend ℓ to $L \cup -L$ in the obvious way. We call $p = l_0 + ... + l_n$ for $l_i \in L \cup -L$ a *path label* from x to y iff some rearrangement of $\ell(l_0), ..., \ell(l_n)$ is a path from x to y.

Notice that if p is a path label from x to y, and q is a path label from y to z, then $p + q$ is a path label from x to z.

The negative answer

Consider the orbit equivalence relation induced by the action $V(L) \curvearrowright \mathcal{P}(V(L) \times X)$ by

$$p \cdot A = \{(q + p, x) \mid (q, x) \in A\}.$$

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$$x \mapsto \{(p, y) \mid p \text{ is the unique path label from } x \text{ to } y\ , y\ E_T\ x\}.$$

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Two problems:

1. $V(L)$ is not a Polish group
2. $\mathcal{P}(V(L) \times X)$ is far from a Polish space.

The negative answer

Problem 1: $V(L)$ is not a Polish group

Instead, we use the free Banach space over L denoted $B(L)$.

Equip $V(L)$ with the mass transportation distance norm (explain visually). Then take $B(L)$ to be the completion with respect to this norm, which is separable as long as L is.

The negative answer

Problem 2: $\mathcal{P}(B(L) \times X)$ **is far from a Polish space.**

Given a Polish space Y, then $\mathcal{F}(Y)$ denotes the space of all closed subsets of Y with the Effros Borel structure. Can be made Polish and natural continuous action $B(L) \curvearrowright \mathcal{F}(B(L) \times X)$.

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Assume $\ell : L \to T$ is *stretched*, which means that $d(p, q) > 1/4$ for any two distinct path labels starting at x. Then

$$x \mapsto \{(p, y) \mid \ell(p) \text{ is the unique path label from } x \text{ to } y \text{ , } y \ E_T \ x\}.$$

is in fact a map from X to $\mathcal{F}(B(L) \times X)$.

THE NEGATIVE ANSWER

Lemma 1

Let T be a locally-finite Borel tree on Polish X. Then there is a stretched Polish edge labeling $\ell : L \to T$.

By fixing a Borel linear order on X and by the fact that every standard Borel space can be made Polish, we have a Polish edge labeling $\ell_0 : L_0 \to T$.

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Lemma 1

Let T be a locally-finite Borel tree on Polish X. Then there is a stretched Polish edge labeling $\ell : L \to T$.

By fixing a Borel linear order on X and by the fact that every standard Borel space can be made Polish, we have a Polish edge labeling $\ell_0 : L_0 \to T$.

For $n \in \omega$, let G^n be the graph on L_0 where l_0, l_1 are adjacent if there is a path of length at most 2^n between $\ell(l_0), \ell(l_1)$.

Each G^n_T is a locally finite Borel graph and thus has a countable proper Borel coloring $c_n : L_0 \to \omega$.

THE NEGATIVE ANSWER

Let τ be a compatible Polish topology on L_0 in which each c_n is continuous.

Let $c : L_0 \to \omega^\omega$ be $c(l) = \langle c_n(l) \mid n \in \omega \rangle$.

The negative answer

Let τ be a compatible Polish topology on L_0 in which each c_n is continuous.

Let $c : L_0 \to \omega^\omega$ be $c(l) = \langle c_n(l) \mid n \in \omega \rangle$.

Now let L be the graph of c equipped with the product topology of τ and the usual topology on Baire space.

Let ℓ be the projection onto the first coordinate composed with ℓ_0.

The negative answer

Given $x, y \in \omega^\omega$, let $d_B(x, y)$ be $1/2^n$ where n is least such that $x(n) \neq y(n)$.

Let d_0 be any compatible complete metric for (L_0, τ). Then let

$$d((x, y), (x', y')) := d_0(x, x') + d_B(y, y')$$

be the metric on L.

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Let d_0 be any compatible complete metric for (L_0, τ). Then let

$$d((x, y), (x', y')) := d_0(x, x') + d_B(y, y')$$

be the metric on L.

Given a path label $p \in B(L)$ of length between 2^n and 2^{n+1} then for any l, l' appearing in the path we have $(l, l') \in G^{n+1}$ and thus $d(l, l') \geq 1/2^{n+1}$.

In particular, $||p|| \geq \frac{1}{2}[2^n \times 1/2^{n+1}] = 1/4$.

The negative answer

Every orbit equivalence relation induced by an abelian Polish group can be lifted to an action of ℓ_1 by Mackey-Hjorth [see e.g. [GP03]]

This completes the proof of the first result.

The positive answer

Theorem 4 (A.)

Any CBER classifiable by $\mathbb{R}^\omega$ is hyperfinite.

Follows from ideas of earlier result (further developing ideas from work of Ding-Gao on non-Archimedean abelian Polish groups)

Theorem 5 (A. [All23])

Any orbit equivalence relation that is $\mathbf{\Pi}^0_3$ and classifiable by non-Archimedean abelian Polish group is Borel-reducible to E_0^ω.

combined with

Theorem 6 (Cotton [Cot19])

Any CBER classifable by a locally compact abelian Polish group is hyperfinite.

The positive answer

Using the Hjorth analysis of Polish group actions, we get

Proposition 1

Any orbit equivalence relation that is $\mathbf{\Pi}^0_3$ and classifiable by a countable product of locally-compact abelian Polish groups is Borel-reducible to E_0^ω.

Then for the final result we just apply the following dichotomy theorem.

Theorem 7 (Hjorth-Kechris [HK97])

If $E \leq_B E_0^\omega$ then either $E \leq_B E_0$ or $E_0^\omega \leq_B E$.

FINAL THOUGHTS

Josh Frisch and Forte Shinko claim that with an additional step one can show that in fact every CBER is classifiable by ℓ_1.

We don't know if there is an equivalence relation classifiable by a TSI Polish group but not by any abelian Polish group.

Preprint

Countable Borel treeable equivalence relations are classifiable by ℓ_1

https://arxiv.org/abs/2305.01049

Thank you!

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